Section 1.1

In math, we use **variables** to talk about

[1] **a thing that you may not know the value of**

(there could be more than one value)

OR [2] **each thing within a certain group of things**

(without naming each thing separately).

ex. Rewrite using variables. Which of the two cases above is each situation ?

"Everyone enrolled in Math 22 passed Math 43."

For every person x who is enrolled in Math 22,

x passed Math 43. (Case [2])

"Some real numbers are smaller than their own square roots."

For some real number y, $y < \sqrt{y}$. (Case [1])

Rewrite without using variables. (Try to make the answer sound like natural language.)

"There is a DeAnza instructor r, such that r's wife is a chef."

There is a DeAnza instructor whose wife is a chef.

"For all positive integers t, $\frac{1}{t} \leq t^2$."

The reciprocal of every positive integer

is less than or equal to that integer's square.

A **universal statement** says that

a certain property is true for every thing within a group.

Universal statements often use the words **“all”, “any” or “every”,**

or the phrases **“for all”, “for any” or “for every”.**

A **conditional statement** says that

if one situation is true, then another situation must also be true.

Conditional statements often use the words **“if”/”then”.**

An **existential statement** says that

there is at least one thing in a group for which a certain property is true.

Existential statements often use the word **“some”,**

or the phrases **“for some”, “there is” or “there exists”.**

ex. Classify each statement.

“Somebody in this class hasn’t signed in yet.” **(EXISTENTIAL)**

“All DeAnza students have a DeAnza ID number.” **(UNIVERSAL)**

“If 97 is odd, then 97^2 is odd.” **(CONDITIONAL)**

A **universal conditional statement** says that

for every thing within a group, if one property is true,

then another property must also be true.

In other words, a universal conditional statement is a statement that is

both universal and conditional.

eg. "For all American citizens p, if p is eligible to vote, then p is at least 18 years old."

Universal conditional statements can be written to appear as either strictly universal or strictly conditional.

eg. Universal: "For all American citizens w who are eligible to vote, w is at least 18 years old."

OR "All American citizens who are eligible to vote are at least 18 years old."

NEW GROUP = THINGS IN ORIGINAL GROUP FOR WHICH THE "IF" PROPERTY IS TRUE

Conditional: "If an American citizen w is eligible to vote, then w is at least 18 years old."

OR "If an American citizen is eligible to vote, he is at least 18 years old."

NEW "IF" PROPERTY = ORIGINAL "IF" PROPERTY ALONG WITH PROPERTY OF BEING IN ORIGINAL GROUP

Universal conditional statements can also be written to appear neither explicitly universal nor explicitly conditional.

eg. "American citizens must be 18 years old to be eligible to vote."

SIDE NOTE: It is possible to write every universal statement as a conditional statement using the method above.

ex. Rewrite using the given structures.

"For all real numbers m, if $m < 0$, then $\sqrt{m}$ is an imaginary number."

USING A VARIABLE:

[a] If m **is a negative real number** , then **√m is an imaginary number.**

[b] For all **negative real numbers** m, **√m is an imaginary number.**

WITHOUT USING A VARIABLE:

[c] All **negative real numbers have imaginary square roots.**

[d] The square root of **any negative real number is imaginary.**

[e] If **a real number is negative** , then **its square root is imaginary.**

ex. Rewrite using the formal universal conditional structure.

“The sine of every acute angle is positive.”

For all **angles *a***, if ***a* is acute**, then **sin *a* > 0.**

A **<u>universal existential statement</u>** says that

for every thing in a group,

there exists at least one other thing in some group

for which a certain property is true.

(NOTE: the second thing may or may not be different from the first thing,

and the second thing may or may not be from the same group as the first thing)

eg. “For every positive number x, there is an acute angle y such that $y = \tan^{-1} x$.”

Universal existential statements can be written in a less formal structure,
which may make the existential portion less explicit by eliminating the second variable or even both variables.

eg. “For all positive numbers x, x has an acute inverse tangent.”

“Every positive number has an acute inverse tangent.”

“All positive numbers have acute inverse tangents.”

“The inverse tangent of a positive number is always acute.”

ex. <u>Rewrite using the given structures.</u>

“For all negative numbers j, there is a positive number k, such that $k = j^2$.”

USING ONE VARIABLE:

[a] For all **negative numbers** *j*, *j* has **a positive square.**

[b] For every **negative number *j*** **,** there is **a positive square for *j*.**

WITHOUT USING A VARIABLE:

[c] All **negative numbers have positive squares.**

[d] The square of **every negative number is positive.**

[e] For every **negative number**, there is **a positive square.**

ex. <u>Rewrite using the formal universal existential structure.</u>

“Everybody loves somebody.”

For every **person *h*** , there is **a person *k*** , such that ***h* loves *k*.**

An **<u>existential universal statement</u>** says that

there exists at least one thing in a group

for which a certain property is true for every thing in some group.

(NOTE: the second group may or may not be the same as the first group)

eg. “There is a positive number e such that, for all real numbers r, $e \times r = r$.”

Existential universal statements can be written in a less formal structure,
by eliminating the second variable or even both variables.
It is, however, hard to make the existential or universal portions less explicit.

eg. “There is a positive number e whose product with any real number is that real number.”

“There is a positive number whose product with every real number is the real number.”

“Some positive number, when multiplied by any real number, gives that real number.”

ex. <u>Rewrite using the given structures.</u>

“There is a class g such that, for every Math 22 student b, b has passed g.”

USING ONE VARIABLE:

[a] There is a class g such that **every Math 22 student has passed *g*.**

WITHOUT USING A VARIABLE:

[b] There is **a class that every Math 22 student has passed.**

[c] Some **class has been passed by every Math 22 student.**

ex. <u>Rewrite using the formal existential universal structure.</u>

“Some monument has been seen by every American tourist visiting Paris.”

There is **a monument *s*** such that, for all **American tourists *f* visiting Paris**, ***f* has seen *s*.**

Section 1.2

A set is a collection or group of elements.

eg. if M = set of all Honda car models

then Fit **is an element of** M ie. **Fit $\in$ M**

and Prius **is not an element of** M ie. **Prius $\notin$ M**

Set roster notation (list of elements)

eg. set of factors of 8 = {1, 2, 4, 8}

set of integers from 5 to 20 = **{5, 6, 7, …, 20}** ***(notice the ellipsis notation)***

set of integers greater than 5 = **{5, 6, 7, …}**

Special sets

R = set of all real numbers

$\mathbf{R}^+$ = set of all *positive* real numbers

Z = set of all integers

$\mathbf{Z}^-$ = set of all *negative* integers

Q = set of all rational numbers (quotients of integers)

$\mathbf{Q}^{\mathbf{nonneg}}$ = set of all *non-negative* rational numbers (zero and all positive rational numbers)

Set equality

INFORMAL DEFINITION:

Given sets A and B, we say A and B are equal, or A = B, **if A and B have the same elements.**

ex. If A = {1, 2, 3} and B = {3, 1, 2}, then A **=** B

If C = {0, 2, 4, 6} and D = {2, 4, 6}, then C $\neq$ D

A set can be an element of another set.

eg. Let K = {a, {b}}

a $\in$ K {b} $\in$ K {a} $\notin$ K b $\notin$ K

Set builder notation (specification of property)

A limitation of set roster notation is that for sets with many elements,
you must either list all the elements,which would be impossible for sets with infinitely many elements,
or you must use ellipsis, but the pattern of the elements may not be obvious

eg. {3, 4, 6, 8, 12, 14, …} **= set of all positive integers which are 1 larger than a prime number**

Given a set S, and a property P which may or may not be true for the individual elements of S, we can define a new set

$$\{x \in S \mid P(x)\} \qquad \textbf{OR} \qquad \{x \in S \mid P \text{ is true for } x\}$$

which consists of exactly those elements of S for which P is true, ie. those elements of S which *satisfy* P.

eg. $\{x \in \mathbf{Z}^+ \mid -3 \le x < 3\} = \mathbf{\{1, 2\}}$

$\{x \in \mathbf{Z} \mid x = 6k \text{ for some integer } k\} = \mathbf{\{0, 6, -6, 12, -12, \ldots\}}$

set of all positive integers which are 1 larger than a prime number = $\{x \in \mathbf{Z}^+ \mid x - 1 \text{ is prime}\}$

set of all perfect squares = $\{x \in \mathbf{R} \mid x = n^2 \text{ for some integer } n\}$ **OR** $\{x \in \mathbf{Z}^{nonneg} \mid x = n^2 \text{ for some integer } n\}$

<u>Subsets</u>

DEFINITION:

Given sets A and B, we say A is a subset of B, or $\mathbf{A \subseteq B}$,

if and only if **for all x, if $x \in A$, then $x \in B$.**

Written more casually, $\mathbf{A \subseteq B}$ if and only if **every element of A is an element of B.**

Other ways of reading $A \subseteq B$: **A is contained in B**

B contains A

NOTE: A is not a subset of B, or $\mathbf{A \not\subseteq B}$,

if and only if **there is an element of A that is not an element of B,**

(or more symbolically) **there is an x such that $x \in A$ and $x \notin B$.**

ex. $\{1, 2\} \subseteq \{0, 1, 2, 3\}$

$\{0, 1, 2, 3\} \not\subseteq \{1, 2\}$

$\{1, 2, 4, 8\} \not\subseteq \{0, 1, 2, 3, 4, 5, 6, 7\}$

$\{x \in \mathbf{Z}^+ \mid x \text{ is prime}\} \not\subseteq \{x \in \mathbf{Z}^+ \mid x \text{ is odd}\}$

$\{4, 7\} \subseteq \{4, 7\}$

$1 \not\subseteq \{1, 2\}$ **but** $\mathbf{1 \in \{1, 2\}}$

$\{2\} \not\subseteq \{1, \{2\}\}$ **but** $\mathbf{\{\{2\}\} \subseteq \{1, \{2\}\}}$ **and** $\mathbf{\{2\} \in \{1, \{2\}\}}$

<u>Ordered Pairs; Cartesian Product of 2 Sets</u>

DEFINITION:

Given elements a, b, c, d, we say (a, b) = (c, d) if and only if **a = c and b = d.**

eg. $(1, 4) \neq (4, 1)$ **although** $\mathbf{\{1, 4\} = \{4, 1\}}$

$(0, 2) = (\sin \pi, \sqrt{4})$ **since** $\mathbf{0 = \sin \pi}$ **and** $\mathbf{2 = \sqrt{4}}$

We can think of ordered pairs as special sets,
where the ordered pair (a, b) corresponds to the set {{a}, {a, b}}.

eg. (1, 4) corresponds to the set **{{1}, {1, 4}}**

(4, 1) corresponds to the set **{{4}, {1, 4}} which is clearly different from the set for (1, 4)**

(2, 2) corresponds to the set **{{2}, {2, 2}} = {{2}, {2}} = {{2}}**

{{2, 5}, {5}} corresponds to the ordered pair **(5, 2)**

DEFINITION:

> Given two sets A and B, the Cartesian product of A and B, or **A × B,**
>
> is **{(a, b) | a ∈ A and b ∈ B}.**

Written more casually, A × B is the set of all ordered pairs where

the first element is from A and the second element is from B.

eg. Given A = {1, 3} and B = {1, 2}

A × B = **{(1, 1), (1, 2), (3, 1), (3, 2)}**

B × A = **{(1, 1), (1, 3), (2, 1), (2, 3)}**

ex. Given R = {0, 4} and T = {a, g, r}

R × R = **{(0, 0), (0, 4), (4, 0), (4, 4)}**

T × R = **{(a, 0), (a, 4), (g, 0), (g, 4), (r, 0), (r, 4)}**

The number of elements in the Cartesian product of 2 sets is

the product of the number of elements in the set.

ex. Given Q × P = {(a, t), (h, h), (h, e), (a, e), (a, h), (h, t)}

P = **{t, h, e}**

Q = **{h, a}**